

EXAMPLES OF NON-COMMUTATIVE CREPANT RESOLUTIONS OF COHEN MACAULAY NORMAL DOMAINS

TONY J. PUTHENPURAKAL

ABSTRACT. Let A be a Cohen-Macaulay normal domain. A non commutative crepant resolution (NCCR) of A is an A -algebra Γ of the form $\Gamma = \text{End}_A(M)$, where M is a reflexive A -module, Γ is maximal Cohen-Macaulay as an A -module and $\text{gldim}(\Gamma)_P = \dim A_P$ for all primes P of A . We give bountiful examples of equi-characteristic Cohen-Macaulay normal local domains and mixed characteristic Cohen-Macaulay normal local domains having NCCR. We also give plentiful examples of affine Cohen-Macaulay normal domains having NCCR.

1. INTRODUCTION

Let A be a Cohen-Macaulay normal domain. Van den Bergh [14] defined a *non-commutative crepant resolution* of A (henceforth NCCR) to be an A -algebra Γ of the form $\Gamma = \text{End}_A(M)$, where M is a reflexive A -module, Γ is maximal Cohen-Macaulay as an A -module and $\text{gldim}(\Gamma)_P = \dim A_P$ for all primes P of A . We should remark that Van den Bergh only defined this for Gorenstein normal domains as this has applications in algebraic geometry. However there are many algebraic reasons for consider this generalization, see [4]. For a nice survey on this topic see [10]. In general, it is subtle to construct NCCR's. In this paper we give bountiful examples of Cohen-Macaulay normal domains having a NCCR.

1.1. Mixed Characteristic case: We now outline in brief our construction. Recall $f \in \mathbb{Z}[X_1, \dots, X_n]$ has content 1 if 1 belongs to the ideal generated by the coefficients of f . We say f is \mathbb{Q} -smooth if $\mathbb{Q}[X_1, \dots, X_n]/(f)$ is a regular ring. For a prime p we say f is smooth mod- p if $\mathbb{Z}_p[X_1, \dots, X_n]/(f)$ is a regular ring. It is well-known that if f is \mathbb{Q} -smooth then is smooth mod- p for infinitely many primes p . Our result is:

Theorem 1.2. *Let (A, \mathfrak{m}) be an excellent normal Cohen-Macaulay local domain of mixed characteristic with perfect residue field $k = A/\mathfrak{m}$ of characteristic $p > 0$. Assume A has a NCCR and that $\dim A \geq 2$. Also assume that A has a canonical module. Let $f \in \mathbb{Z}[X_1, \dots, X_n]$ be of content 1. Also assume that f is \mathbb{Q} -smooth*

Date: June 11, 2014.

1991 Mathematics Subject Classification. 14B05, 14A22, 14E15, 13C14, 16E10 .

Key words and phrases. non-commutative crepant resolutions, normal domains, Hensel rings.

and is smooth mod- p . Set $T = A[X_1, \dots, X_n]/(f)$ and let \mathfrak{n} be a maximal ideal of T containing $\mathfrak{m}T$. Set $A(f) = T_{\mathfrak{n}}$. Then

- (i) $A(f)$ is flat over A with regular fiber. In particular if A is Gorenstein then so is $A(f)$.
- (ii) $A(f)$ is an excellent normal Cohen-Macaulay local domain of mixed characteristic with perfect residue field.
- (iii) $A(f)$ has a NCCR.

Furthermore if $\Gamma = \text{Hom}_A(M, M)$ is a NCCR of A then $\Lambda = \Gamma \otimes_A A(f)$ is a NCCR of $A(f)$.

1.3. Two dimensional rings of finite representation type have a NCCR. (see [9, Theorem-6]). For examples of two dimensional mixed characteristic rings of finite representation type see [12]. Using the above recipe we can construct plentiful examples of Cohen-Macaulay local domain of mixed characteristic having NCCR's. If k is algebraically closed then it can be easily shown that if $A(f) \cong A(g)$ as A -algebra's then the hypersurfaces defined by f and g in $\mathbb{A}^n(k)$ are birational

1.4. Equi-characteristic case (local): Let (A, \mathfrak{m}) be an excellent equi-characteristic Cohen-Macaulay local domain with perfect residue field k . Assume A contains k , $\dim A \geq 2$ and that it has a canonical module. Let $f \in k[X_1, \dots, X_n]$ be smooth, i.e., $k[X_1, \dots, X_n]/(f)$ is a regular ring. We show

Theorem 1.5. (with hypotheses as in 1.4) Assume A has a NCCR. Set $T = A[X_1, \dots, X_n]/(f)$. Let \mathfrak{n} be a maximal ideal of T containing $\mathfrak{m}T$. Set $A(f) = T_{\mathfrak{n}}$. Then

- (i) $A(f)$ is flat over A with regular fiber. In particular if A is Gorenstein then so is $A(f)$.
- (ii) $A(f)$ is an equi-characteristic excellent normal Cohen-Macaulay local domain with perfect residue field.
- (iii) $A(f)$ has a NCCR.

Furthermore if $\Gamma = \text{Hom}_A(M, M)$ is a NCCR of A then $\Lambda = \Gamma \otimes_A A(f)$ is a NCCR of $A(f)$.

1.6. In both Theorems it is clear that $M \otimes_A A(f)$ is reflexive and Λ is maximal Cohen-Macaulay as a $A(f)$ -module. To prove finiteness of global dimension of Λ we may complete $A(f)$ (see [13, 1.4]). The essential point is to prove the following result:

Theorem 1.7. Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat local homomorphism of Henselian local rings with fiber $F = B/\mathfrak{m}B$ regular local. Assume the residue fields $k = A/\mathfrak{m}$ and $l = B/\mathfrak{n}$ are perfect. Let M be a finitely generated A -module such that $\Gamma = \text{Hom}_A(M, M)$ has finite global dimension. Then

$$\text{gldim } \Gamma \otimes_A B \leq \text{gldim } \Gamma + \dim F.$$

1.8. Equi-characteristic case (global): Let k be a perfect field. Let A be an affine k -algebra. Assume A is a Cohen-Macaulay normal domain of dimension $d \geq 2$. Suppose A has a NCCR $\Gamma = \text{Hom}_A(M, M)$. Let $f \in k[X_1, X_2, \dots, X_n]$ be a polynomial such that its homogenization \tilde{f} defines a smooth hypersurface in $\mathbb{P}^n(\bar{k})$ (here \bar{k} is the algebraic closure of k). We show

Theorem 1.9. *(with hypotheses as in 1.8) Let $T(f) = A[X_1, \dots, X_n]/(f)$. Then*

- (i) *$T(f)$ is a Cohen-Macaulay normal domain. If A is Gorenstein then so is $T(f)$.*
- (ii) *$\Gamma \otimes_A T(f)$ is a NCCR of $T(f)$.*

Let K be the quotient field of A . It is easily shown that if $T(f) \cong T(g)$ as A -algebras then $V(f)$ is isomorphic to $V(g)$ in the affine space $\mathbb{A}^n(\bar{K})$. Thus there are lot of examples of non-isomorphic rings $A(f)$ having NCCR.

1.10. The main technical tool in this paper is a notion we call absolutely indecomposable modules over a Hensel local ring. Let (A, \mathfrak{m}) be a Henselian local ring of dimension $d \geq 0$ and residue field k . As A is Henselian, the category of finitely generated A -modules is Krull-Schmidt, i.e., any finitely generated A -module is uniquely a finite direct sum of indecomposable A -modules. Let M be a finitely generated A -module and let $\text{rad End}_A(M)$ be the radical of $\text{End}_A(M)$. Recall that a module E is indecomposable if and only if $\text{End}_A(E)$ is local; equivalently $\text{End}_A(E)/\text{rad End}_A(E)$ is a division ring. We say E is *absolutely indecomposable* if $\text{End}_A(E)/\text{rad End}_A(E) \cong k$. If $M = M_1^{a_1} \oplus \dots \oplus M_n^{a_n}$ with M_i absolutely indecomposable then

$\text{End}_A(M)/\text{rad End}_A(M)$ is a direct product of matrix rings over k . This enables us to keep track of $\text{End}_A(M) \otimes B$ when B is flat over A . The main technical result of this paper is:

Theorem 1.11. *Let (A, \mathfrak{m}) be a Henselian local ring with perfect residue field k . Let M be a finitely generated A -module. Then there exists a finite flat extension of the form $R = A[X]/(\phi(X))$ where $\phi(X)$ is monic and $\overline{\phi(X)}$ is irreducible in $k[X]$ such that the R -module $M \otimes_A R$ is a finite direct sum of absolutely indecomposable R -modules. Furthermore $\text{gldim End}_A(M) = \text{gldim End}_R(M \otimes_A R)$.*

We now describe in brief the contents of this paper. In section two we discuss some preliminaries that we need. In section three we introduce the notion of absolutely indecomposable modules. We prove Theorem 1.11 in section 4. In section five we give a description of $\text{End}_A(E)/\text{rad End}_A(E)$. We prove Theorem 1.7 in section 6. In section seven we prove Theorems 1.2 and 1.5. Finally in section eight we prove Theorem 1.9.

2. PRELIMINARIES

In this paper all commutative rings considered are Noetherian. Commutative rings will be denoted as A, B etc. All non-commutative rings considered will be an A -algebra for some commutative Noetherian ring A , furthermore they will be finitely generated as an A -module. Thus all non-commutative rings in this paper will be both left and right Noetherian. Non-commutative rings will be denoted as Γ, Λ etc. Also all modules in this paper are left modules and they will be finitely generated.

In this section we collect some preliminaries which we need. I think that all the results here are already known. I include proofs of some of them as I do not have a reference.

2.1. Suppose Γ is a ring finitely generated over A . Let $\mathfrak{r} = \text{rad } \Gamma$. If A is local with maximal ideal \mathfrak{m} then Γ/\mathfrak{r} is semisimple and $\mathfrak{r} \supseteq \mathfrak{m}\Gamma \supseteq \mathfrak{r}^n$ for some $n \geq 1$. [7, 20.6].

The following is well-known.

Proposition 2.2. *Let (A, \mathfrak{m}) be local and let M be an A -module. Set $\Gamma = \text{End}_A(M)$ and let $\mathfrak{r} = \text{rad } \Gamma$. Then*

- (1) $\text{Hom}_A(M, \mathfrak{m}M)$ is a two sided ideal in Γ .
- (2) $\text{Hom}_A(M, \mathfrak{m}M) \subseteq \mathfrak{r}$.
- (3) $\mathfrak{m}\Gamma \subseteq \text{Hom}_A(M, \mathfrak{m}M)$.

An easy consequence of the above result is

Proposition 2.3. *Let $f: (A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat local map. Let M be an A -module. Set $\Gamma = \text{End}_A(M)$ and let $\mathfrak{r} = \text{rad } \Gamma$. Set $\Lambda = \Gamma \otimes_A B$. Then $\mathfrak{r} \otimes B$ is a two sided ideal contained in $\text{rad } \Lambda$.*

Proof. Put $\mathfrak{q} = \mathfrak{r} \otimes B$. Clearly \mathfrak{q} is a two sided ideal of Λ . If we prove $\mathfrak{q}^n \subseteq \text{rad } \Lambda$ for some n then we are done for $\Lambda/\text{rad } \Lambda$ is semisimple.

By 2.1 we have that $\mathfrak{r}^n \subseteq \mathfrak{m}\Gamma$ for some $n \geq 1$. Let $\phi_1 \otimes b_1, \phi_2 \otimes b_2, \dots, \phi_n \otimes b_n \in \mathfrak{q}$. Set

$$\begin{aligned} \psi &= \phi_1 \otimes b_1 \circ \phi_2 \otimes b_2 \circ \dots \circ \phi_n \otimes b_n, \\ &= (\phi_1 \circ \phi_2 \circ \dots \circ \phi_n) \otimes (b_1 b_2 \dots b_n) \end{aligned}$$

Thus $\psi = \phi \otimes b$ for some $\phi \in \mathfrak{r}^n$ and $b \in B$. As $\mathfrak{r}^n \subseteq \mathfrak{m}\Gamma$ we get that $\phi(M) \subseteq \mathfrak{m}M$. So $\psi(M \otimes B) \subseteq \mathfrak{m}(M \otimes B)$. It follows that $\psi \in \text{Hom}_B(M \otimes B, \mathfrak{n}(M \otimes B)) \subseteq \text{rad } \Lambda$ (by 2.2). It follows that $\mathfrak{q}^n \subseteq \text{rad } \Lambda$. \square

A natural question is when $\mathfrak{r} \otimes B = \text{rad } \Lambda$? We prove

Lemma 2.4. *Let $\phi: (A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat local map with $\mathfrak{m}B = \mathfrak{n}$. Assume $k = A/\mathfrak{m}$ is perfect. Let M be an A -module. Set $\Gamma = \text{End}_A(M)$ and let $\mathfrak{r} = \text{rad } \Gamma$. Set $\Lambda = \Gamma \otimes_A B$. Then $\mathfrak{r} \otimes B = \text{rad } \Lambda$.*

Proof. By 2.3 we have that $\mathfrak{r} \otimes B \subseteq \text{rad } \Lambda$. It suffices to show that $(\Gamma/\mathfrak{r}) \otimes B$ is semisimple.

By 2.1 we have that Γ/\mathfrak{r} is semisimple. So

$$\Gamma/\mathfrak{r} = M_{n_1}(\Xi_1) \times \cdots \times M_{n_r}(\Xi_r)$$

where Ξ_1, \dots, Ξ_r are division algebras over k . Also note that $k \subseteq Z(\Gamma/\mathfrak{r})$, the center of Γ/\mathfrak{r} . It follows that $k \subseteq Z(\Xi_i)$ for each $i = 1, \dots, r$.

It suffices to show that $M_n(\Xi) \otimes B$ is semisimple where Ξ is a division algebra finite dimensional over k and $k \subseteq F = Z(\Xi)$. Set $l = B/\mathfrak{n}$. As k is perfect F is separable over k and so $F \otimes_k l = K_1 \times \cdots \times K_s$ where K_i are finite field extensions of l . Notice

$$\begin{aligned} M_n(\Xi) \otimes_A B &= M_n(\Xi) \otimes_k k \otimes_A B \\ &= M_n(\Xi) \otimes_k l \\ &= M_n(\Xi) \otimes_F F \otimes_k l \\ &= M_n(\Xi) \otimes_F (K_1 \times K_2 \times \cdots \times K_s) \\ &= (M_n(\Xi) \otimes_F K_1) \times (M_n(\Xi) \otimes_F K_2) \times \cdots \times (M_n(\Xi) \otimes_F K_s). \end{aligned}$$

It suffices to show that $M_n(\Xi) \otimes_F K$ is semisimple where K is an extension of F . We first note that by [7, 15.1], the ring $\Upsilon = \Xi \otimes_F K$ is a simple ring. Also note that $K = 1 \otimes K$ is a subring of Υ . As Ξ is finite dimensional over F we get that Υ is finite dimensional as a K -vector space. In particular Υ is Artinian. Thus by [7, 3.1] Υ is a semisimple ring. Finally notice that as Ξ is finite dimensional over F , the natural ring homomorphism $M_n(\Xi) \otimes_F K \rightarrow M_n(\Xi \otimes_F K)$ is an isomorphism, see [7, 7.4]. The result follows. \square

An easy consequence of the above result is the following:

Corollary 2.5. *(with hypotheses as in 2.4)*

$$\text{gldim } \Gamma = \text{gldim } \Lambda.$$

Proof. For a left Γ module M let $\text{projdim}_\Gamma M$ denote its projective dimension. By an argument similar to [13, 1.1] we can show

$$\text{projdim}_\Gamma M = \text{projdim}_\Lambda M \otimes_A B$$

By [13, 1.3] we have that $\text{gldim } \Gamma = \text{projdim } \Gamma/\mathfrak{r}$ (here we consider \mathfrak{r} as a left Γ -ideal). Similarly $\text{gldim } \Lambda = \text{projdim } \Lambda/\text{rad } \Lambda$. By 2.4 we have that

$$\frac{\Gamma}{\mathfrak{r}} \otimes_A B = \frac{\Lambda}{\text{rad } \Lambda}.$$

The result follows. \square

3. ABSOLUTELY INDECOMPOSABLE MODULES

Let (A, \mathfrak{m}) be a Henselian local ring of dimension $d \geq 0$ and residue field k . Let M be an A -module and let $\text{rad End}_A(M)$ be the radical of $\text{End}_A(M)$. Recall that a module E is indecomposable if and only if $\text{End}_A(E)$ is local; equivalently $\text{End}_A(E)/\text{rad End}_A(E)$ is a division ring. We say E is *absolutely indecomposable* if $\text{End}_A(E)/\text{rad End}_A(E) \cong k$.

We need the following:

Definition 3.1. Suppose (A, \mathfrak{m}) is a Henselian local ring of dimension $d \geq 0$ and residue field k . Let $K \supseteq k$ be a field. Then $\mathcal{F}(A, K)$ is the collection of Henselian local rings (B, \mathfrak{n}) such that

- (1) there is a flat local map $\phi: A \rightarrow B$.
- (2) $\mathfrak{m}B = \mathfrak{n}$.
- (3) $B/\mathfrak{n} \cong K$ over k .

Note that by [1, App. Théorème 1, Corollaire] there exists a local ring B (not-necessarily Henselian) satisfying (1), (2) and (3). Clearly $\hat{B} \in \mathcal{F}(A, K)$.

The word "absolutely indecomposable" is well chosen thanks to the following result.

Theorem 3.2. Let (A, \mathfrak{m}) be a Henselian local ring with perfect residue field k . Let M be an A -module. The following are equivalent:

- (i) M is absolutely indecomposable.
- (ii) $M \otimes_A B$ is absolutely indecomposable for every $B \in \mathcal{F}(A, K)$ for any extension field K of k .
- (iii) $M \otimes_A B$ is indecomposable for every $B \in \mathcal{F}(A, K)$ for any extension field K of k .
- (iv) $M \otimes_A B$ is indecomposable for some $B \in \mathcal{F}(A, K)$ with K algebraically closed.

Proof. Set $\text{End}_A(M) = \Gamma$ and $\mathfrak{r} = \text{rad } \Gamma$. For $B \in \mathcal{F}(A, K)$ set $\Lambda = \Gamma \otimes B$ and $\mathfrak{q} = \mathfrak{r} \otimes B$.

(i) \implies (ii). We have the exact sequence $0 \rightarrow \mathfrak{r} \rightarrow \Gamma \rightarrow k$. Tensoring with B yields

$$(*) \quad 0 \rightarrow \mathfrak{q} \rightarrow \Lambda \rightarrow K \rightarrow 0.$$

By 2.4 we have that $\mathfrak{q} = \text{rad } \Lambda$. It follows that $M \otimes B$ is absolutely indecomposable.

(ii) \implies (iii). Clear.

(iii) \implies (iv). Clear.

(iv) \implies (i). Notice $\Lambda/\text{rad } \Lambda$ is a division algebra which is finite over K . As K is algebraically closed we get that $\Lambda/\text{rad } \Lambda = K$.

As $M \otimes B$ is indecomposable we have that M is also indecomposable. Say $\Gamma/\mathfrak{r} = \Xi$ where Ξ is a division ring. Say $\dim_k \Xi = r$. By 2.4 we have that $\mathfrak{q} = \text{rad } \Lambda$.

It follows that $\Xi \otimes_k K = K$. Computing dimensions as vector space over K we get that $r = 1$. So $\Xi = k$. Thus M is absolutely indecomposable. \square

4. PROOF OF THEOREM 1.11

In this section we give a proof of Theorem 1.11. The essential ingredient is a construction which we now describe:

4.1. Let (A, \mathfrak{m}) be a Henselian local ring with perfect residue field k . Let \bar{k} be the algebraic closure of k . Let

$$\mathcal{C}_k = \{E \mid E \text{ is a finite extension of } k, \text{ and } E \subseteq \bar{k}\}.$$

Order \mathcal{C}_k with the inclusion as partial order. Note that \mathcal{C}_k is a directed set, for if $E, F \in \mathcal{C}_k$ then the composite field $EF \in \mathcal{C}_k$ and clearly $EF \supseteq E$ and $EF \supseteq F$. We prove

Theorem 4.2. *(with hypotheses as in 4.1) There exists a direct system of local rings $\{(A^E, \mathfrak{m}^E) \mid E \in \mathcal{C}_k\}$ such that*

- (1) *A^E is a finite flat extension with $\mathfrak{m}A^E = \mathfrak{m}^E$. Furthermore $A^E/\mathfrak{m}^E \cong E$ over k .*
- (2) *A^E is Henselian.*
- (3) *For any $F, E \in \mathcal{C}_k$ with $F \subseteq E$ the maps in the direct system $\theta_F^E: A^F \rightarrow A^E$ is flat and local with $\mathfrak{m}^F A^E = \mathfrak{m}^E$.*

The ring $T = \lim_{E \in \mathcal{C}_k} A^E$ will have nice properties which enables us to prove Theorem 1.11.

4.3. Construction-1.1: For every $E \in \mathcal{C}_k$ we construct a ring A^E as follows. As k is perfect, E is a separable extension of k . So by primitive element theorem $E = k(\alpha_E)$ for some $\alpha_E \in E$. Let

$$p_E(X) = p_{E, \alpha_E}(X) = \text{Irr}(\alpha_E, k),$$

be the unique monic minimal polynomial of α_E over k . Let $f_E(X) = f_{E, \alpha_E}(X)$ be a monic polynomial in $A[X]$ such that $\overline{f_E(X)} = p_E(X)$. Set

$$A^E = \frac{A[X]}{(f_E(X))}.$$

Our construction of course depends on choice of α_E and the choice of $f_E(X)$. We will simply fix one choice of α_E and $f_E(X)$. We prove:

Proposition 4.4. *(with hypotheses as in 4.3)*

- (i) *A^E is a finite flat extension of A .*
- (ii) *A^E has a unique maximal ideal \mathfrak{m}^E . Furthermore*
 - (a) *$\mathfrak{m}A^E = \mathfrak{m}^E$.*

- (b) $A^E/\mathfrak{m}^E \cong E$.
- (iii) A^E is a Henselian local ring.

Proof. (i) This is clear.

(ii) Notice

$$A^E/\mathfrak{m}A^E = k[X]/(p_E(X)) \cong E.$$

It follows that $\mathfrak{m}A^E$ is a maximal ideal of A^E . Also as A^E is a finite extension of A any maximal ideal \mathfrak{n} of A^E will contain \mathfrak{m} . It follows that $\mathfrak{m}^E = \mathfrak{m}A^E$ is the unique maximal ideal of A^E . Clearly (a), (b) hold.

(iii). Let S be a finite A^E -algebra. Then note that S is a finite A -algebra. As A is Henselian we get that S is a product of local rings. Thus A^E is Henselian. \square

4.5. Construction-1.2: Let $k \subseteq F \subseteq E$ be a tower of fields. We construct a ring homomorphism $\theta_F^E: A^F \rightarrow A^E$ as follows: Notice $\alpha_F \in E$. It follows that $P_F(Y) = (Y - \alpha_F)h(Y)$ for some polynomial $h(Y) \in E[Y]$. As F is separable over k , the roots of $P_F(Y)$ are all distinct. It follows that $(Y - \alpha_F, h(Y)) = 1$. The ring A^E is Henselian. So there exists $\xi \in A^E$ and $g(Y) \in A^E[Y]$ such that $\bar{\xi} = \alpha_F$, $\overline{g(Y)} = h(Y)$ and $f_F(Y) = (Y - \xi)g(Y)$.

Claim-1: If $\xi' \in A^E$ such that $f_F(\xi') = 0$ and $\bar{\xi'} = \alpha_F$ then $\xi' = \xi$.

Note that $0 = f_F(\xi') = (\xi' - \xi)g(\xi')$. As $h(\alpha_F) \neq 0$ we get that $g(\xi')$ is a unit in A^E . So $\xi' = \xi$.

Notation: Denote ξ as ξ_F^E .

Define

$$\begin{aligned} \theta_F^E: A^F &\rightarrow A^E, \\ a &\rightarrow a \text{ for all } a \in A, \\ X &\rightarrow \xi_F^E \end{aligned}$$

We prove:

Proposition 4.6. (with hypotheses as in 4.5)

- (i) θ_F^E is a homomorphism of A -algebra's.
- (ii) θ_F^E is a local map and $\mathfrak{m}^F A^E = \mathfrak{m}^E$.
- (iii) A^E is a flat A^F -module (via θ_F^E).
- (iv) If $k \subseteq F \subseteq E \subseteq L$ is a tower of fields then we have a commutative diagram

$$\begin{array}{ccc} A^F & & \\ \theta_F^E \downarrow & \searrow \theta_F^L & \\ A^E & \xrightarrow{\theta_E^L} & A^L \end{array}$$

Proof. (i) This is clear since $\theta_F^E(a) = a$ for each $a \in A$.

(ii). As θ_F^E fixes A we get that $\theta_F^E(\mathfrak{m}^F) = \theta_F^E(\mathfrak{m}A^F) \subseteq \mathfrak{m}A^E = \mathfrak{m}^E$. Thus θ_F^E is local.

Also note that $\mathfrak{m}_F A^E = \mathfrak{m} A^F A^E = \mathfrak{m} A^E = \mathfrak{m}^E$.

(iii). Suppose $\dim_F E = r$ and $\dim_k F = s$. Then $\dim_k E = rs$. Notice $A^E/\mathfrak{m}^F A^E = E$ it follows that A^E is generated as an A^F module by r -elements; say $\{e_1, \dots, e_r\}$. Similarly A^F is generated as an A -module by s elements; say $\{f_1, \dots, f_s\}$. It follows that A^E is generated over A by $\{e_i f_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$. However A^E is a free A -module of rank $\dim_k E = rs$. It follows that $\{e_i f_j\}_{i,j}$ is a basis of the A -module A^E .

Claim: $\{e_1, e_2, \dots, e_r\}$ is a basis of A^E over A^F .

We already have that A^E is generated as an A^F module by $\{e_1, e_2, \dots, e_r\}$. Suppose

$$\beta_1 e_1 + \beta_2 e_2 + \dots + \beta_r e_r = 0 \quad \text{for some } \beta_i \in A^F.$$

Write

$$\beta_i = \sum_{j=1}^s \gamma_{ij} f_j \quad \text{for some } \gamma_{ij} \in A.$$

It follows that

$$\sum_{i,j} \gamma_{ij} e_i f_j = 0$$

As $\{e_i f_j\}_{i,j}$ is a basis of the A -module A^E we get that $\gamma_{ij} = 0$ for all i, j . It follows that $\beta_i = 0$ for all i . Thus $\{e_1, e_2, \dots, e_r\}$ is a basis of A^E over A^F .

(iv). Note $f_F(Y) = (Y - \xi_F^E)g(Y)$ in $A^E[Y]$. Applying θ_F^E and noting that it fixes A we get

$$f_F(Y) = (Y - \theta_F^E(\xi_F^E))\theta(g(Y)) \quad \text{in } A^L[Y].$$

Notice $\overline{\theta_F^E(\xi_F^E)} = \alpha_F$. So by uniqueness we get

$$\xi_F^L = \theta_F^E(\xi_F^E).$$

The result follows. □

As a consequence we get

Proof of Theorem 4.2. This follows from 4.4 and 4.6. □

4.7. Construction-1.3: Set

$$T = \lim_{E \in \mathcal{C}_k} A^E,$$

and let $\theta_E: E \rightarrow T$ be the maps such that for any $F \subseteq E$ in \mathcal{C}_k we have $\theta_E \circ \theta_F^E = \theta_F$.

For $F \in \mathcal{C}_k$ set

$$\mathcal{C}_F = \{E \mid E \text{ is a finite extension of } F\}.$$

Then clearly \mathcal{C}_F is cofinal in \mathcal{C}_k . Thus we have

$$T = \lim_{E \in \mathcal{C}_F} A^E.$$

We have the following properties of T .

Theorem 4.8. (with hypotheses as in 4.7)

(i) T is a Noetherian ring.

- (ii) T is a flat A -module.
- (iii) T is a flat A^F -module for any $F \in \mathcal{C}_k$.
- (iv) The map θ_E is injective for any $E \in \mathcal{C}_k$.
- (v) By (iv) we may write $T = \bigcup_{E \in \mathcal{C}_k} A^E$. Set $\mathfrak{m}^T = \bigcup_{E \in \mathcal{C}_k} \mathfrak{m}^E$. Then \mathfrak{m}^T is the unique maximal ideal of T .
- (vi) $\mathfrak{m}T = \mathfrak{m}^T$.
- (vii) $T/\mathfrak{m}^T \cong \bar{k}$.
- (viii) T is a Henselian ring.

Proof. (i). As $A^F \rightarrow A^E$ is flat whenever $F \subseteq E$ and $\mathfrak{m}^F A^E = \mathfrak{m}^E$ we get that T is Noetherian, see [3, Chap.0, (10.3.13)].

(ii). Let N be any A -module. Notice for any $i \geq 1$ we have

$$\begin{aligned} \mathrm{Tor}_i^A(T, N) &= \mathrm{Tor}_i^A\left(\lim_{E \in \mathcal{C}_k} A^E, N\right), \\ &= \lim_{E \in \mathcal{C}_k} \mathrm{Tor}_i^A(A^E, N) \\ &= 0. \end{aligned}$$

Thus T is a flat as an A -module.

(iii). This is similar to (ii).

(iv). This follows since each map θ_F^E in the direct limit is injective.

(v). It is clear that \mathfrak{m}^T is an ideal in T . Suppose $\xi \notin \mathfrak{m}^T$. Then $\xi \notin \mathfrak{m}^E$ for some E . This implies that ξ is a unit in A^E . So ξ is a unit in T . Thus \mathfrak{m}^T is the unique maximal ideal of T .

(vi). Clearly $\mathfrak{m}T \subseteq \mathfrak{m}^T$. Let $\xi \in \mathfrak{m}^T$. Then $\xi \in \mathfrak{m}^E$ for some $E \in \mathcal{C}_k$. But $\mathfrak{m}^E = \mathfrak{m}A^E$. It follows that $\xi \in \mathfrak{m}T$. Thus $\mathfrak{m}T = \mathfrak{m}^T$.

(vii). The inclusion $\theta^E: A^E \rightarrow T$ is flat local map of A -algebras and so induces an inclusion of fields $\overline{\theta^E}: E \rightarrow T/\mathfrak{m}^T$ over k . It follows that $L = T/\mathfrak{m}^T$ contains \bar{k} . Let $\bar{\xi} \in L$. Let $\xi \in T$ be its pre-image. Say $\xi \in A^E$. Then notice the map $\overline{\theta^E}$ maps $\xi + \mathfrak{m}^E$ to $\bar{\xi}$. It follows that $L = \bar{k}$.

(viii). Let $f(Y) \in T[Y]$ be a monic polynomial such that its residue class $\overline{f(Y)}$ modulo $\mathfrak{m}^T T[Y]$ has a factorization $\overline{f} = g'h'$ with monic polynomials $g', h' \in T/\mathfrak{m}^T[Y]$ and $(g', h') = 1$. By (v) there exists $E \in \mathcal{C}_k$ such that $f(Y) \in A^E[Y]$. We may assume that all coefficients of $g', h' \in F$ for some $F \in \mathcal{C}_k$. Set $K = EF$. Then note that $f \in A^K[Y]$ and modulo $\mathfrak{m}^K K[Y]$ we have a factorization $\overline{f} = g'h'$. As A^K is Henselian we have that there exists monic polynomials $g, h \in A^K[Y]$ with $f = gh$ and $\overline{g} = g'$ and $\overline{h} = h'$. Now note that $g, h \in T[Y]$. \square

The significance of T is that certain crucial properties descend to a finite extension E of k .

Lemma 4.9. *(with hypotheses as above)*

- (1) Let M be a T -module. Then there exists $E \in \mathcal{C}_k$ and an A^E -module N such that $M = N \otimes_{A^E} T$.
- (2) Let N_1, N_2 be A^E -modules for some $E \in \mathcal{C}_k$. Suppose there is a T -linear map $f: N_1 \otimes_{A^E} T \rightarrow N_2 \otimes_{A^E} T$. Then there exists $K \in \mathcal{C}_k$ with $K \supseteq E$ and an A^K -linear map $g: N_1 \otimes_{A^E} A^K \rightarrow N_2 \otimes_{A^E} A^K$ such that $f = g \otimes T$. Furthermore if f is an isomorphism then so is g .

Proof. (1) Let $F_1 \xrightarrow{\phi} F_0 \rightarrow M \rightarrow 0$ be a finite presentation of M . Say $\phi = (a_{ij})$. Then by 4.8-(v) there exists $E \in \mathcal{C}_k$ such that all $a_{ij} \in A^E$. Consider a presentation $G_1 \xrightarrow{\phi} G_0 \rightarrow N \rightarrow 0$ of A^E -module N . Clearly $N \otimes_{A^E} T \cong M$.

(2). Notice $\text{Hom}_T(N_1 \otimes_{A^E} T, N_2 \otimes_{A^E} T) \cong \text{Hom}_{A^E}(N_1, N_2) \otimes_{A^E} T$. Thus

$$f = f_1 \otimes \xi_1 + \cdots + f_s \otimes \xi_s, \quad \text{for some } f_i \in \text{Hom}_{A^E}(N_1, N_2) \text{ and } \xi_i \in T.$$

Then by 4.8-(v) there exists $F \in \mathcal{C}_k$ such that all $\xi_i \in A^F$. Let $K = EF$. Set

$$g = f_1 \otimes \xi_1 + \cdots + f_s \otimes \xi_s \in \text{Hom}_{A^E}(N_1, N_2) \otimes_{A^E} A^K.$$

Clearly $g \otimes T = f$.

Let U, V be the kernel and cokernel of g . If f is an isomorphism then $U \otimes_{A^K} T = V \otimes_{A^K} T = 0$. By 4.8-(iii) we have that T is a faithfully flat extension of A^K . It follows that $U = V = 0$. Thus g is an isomorphism. \square

We now give

Proof of Theorem 1.11. We make the construction as in 4.2. Let $M \otimes_A T = L_1^{r_1} \otimes \cdots \otimes L_m^{r_m}$ where L_1, \dots, L_m are indecomposable T -modules. By 4.9 it follows that there exists $E \in \mathcal{C}_k$ and A^E -modules N_i with $N_i \otimes T = L_i$ for $i = 1, \dots, m$. By 3.2 it follows that N_i are absolutely indecomposable. Notice

$$M \otimes_A T = (M \otimes_A A^E) \otimes_{A^E} T \cong \left(\bigoplus_{i=1}^s N_i^{r_i} \right) \otimes_{A^E} T.$$

By 4.9 there exists $K \in \mathcal{C}_k$ with $K \supseteq E$ such that

$$M \otimes_{A^K} A^K = (M \otimes_A A^E) \otimes_{A^E} A^K \cong \left(\bigoplus_{i=1}^s N_i^{r_i} \right) \otimes_{A^E} A^K = \left(\bigoplus_{i=1}^s N_i^{r_i} \otimes_{A^E} A^K \right).$$

By 3.2 the A^K -modules $N_i \otimes_{A^E} A^K$ are absolutely indecomposable. We take $R = A^K$. Note that R has the required form by 4.3. By 2.5 we get that $\text{gldim End}_A(M) = \text{gldim End}_R(M \otimes_A R)$. \square

The following result will be useful later.

Lemma 4.10. *Let $\phi: (A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat local map of Henselian rings. Assume the residue fields k, l of A and B are perfect. Let M be an A module. Then*

there exists a commutative diagram of flat local maps of Henselian rings

$$\begin{array}{ccc} A & \xrightarrow{\eta} & A' \\ \downarrow \phi & & \downarrow \phi' \\ B & \xrightarrow{\delta} & B' \end{array}$$

such that

- (1) (A', \mathfrak{m}') is a finite flat extension of A with $\mathfrak{m}A' = \mathfrak{m}'$.
- (2) $M \otimes A'$ is a direct sum of absolutely indecomposable A' -modules.
- (3) (B', \mathfrak{n}') is a finite flat extension of B with $\mathfrak{n}B' = \mathfrak{n}'$.
- (4) If the fiber F of ϕ is regular local then so is the fiber F' of ϕ' .

Proof. (1) and (2): Let $A' = A[X]/(f(X))$ be as in Theorem 1.11.

(3). Set $\tilde{B} = B \otimes A'$. Then clearly \tilde{B} is a flat extension of A' . Furthermore \tilde{B} is a flat extension of B . Also \tilde{B} is a finite extension of B . As B is Henselian we get that \tilde{B} is a direct product of local rings say $\tilde{B} = B_1 \times \cdots \times B_s$. We note that as k is perfect we get that

$$\tilde{B} \otimes_B B/\mathfrak{n} \cong k[X]/(f(X)) \otimes_k l \cong K_1 \times \cdots \times K_m$$

where K_1, \dots, K_m are finite field extensions of l . We now note that B_1 is localization of \tilde{B} at a maximal ideal. As l is perfect we get that B_1 is a finite separable extension of B . So $\mathfrak{n}B$ is the maximal ideal of B_1 , see [15, 2.5]. Set $B' = B_1$.

(4). We note that δ induces a flat map $\bar{\delta}: B/\mathfrak{m}B \rightarrow B'/\mathfrak{m}B'$. We note that as $\mathfrak{m}A' = \mathfrak{m}'$ we get that $\mathfrak{m}B' = \mathfrak{m}'B'$. Thus $\bar{\delta}: F \rightarrow F'$ is a flat map. As $\mathfrak{n}B' = \mathfrak{n}'$ we get that the fiber of $\bar{\delta}$ is a field. So if F is regular then so is F' , see [11, 23.7]. \square

5. A DESCRIPTION OF $\text{End}_A(E)/\text{rad End}_A(E)$

In this section (A, \mathfrak{m}) is a Henselian ring and E is a finitely generated A -module. Assume $E = E_1^{a_1} \oplus \cdots \oplus E_s^{a_s}$ where E_1, E_2, \dots, E_s are mutually non-isomorphic indecomposable A -modules. We prove

Theorem 5.1. *(with hypotheses as above)*

$$\frac{\text{End}_A(E)}{\text{rad End}_A(E)} \cong M_{a_1} \left(\frac{\text{End}_A(E_1)}{\text{rad End}_A(E_1)} \right) \times \cdots \times M_{a_s} \left(\frac{\text{End}_A(E_s)}{\text{rad End}_A(E_s)} \right)$$

(here for a ring Γ , we denote by $M_n(\Gamma)$ the ring of $n \times n$ matrices over Γ).

5.2. Let M, N be two A -modules. Then note that we have an isomorphism of rings

$$\text{End}_A(M \oplus N) \cong \begin{bmatrix} \text{End}_A(M) & \text{Hom}_A(N, M) \\ \text{Hom}_A(M, N) & \text{End}_A(N) \end{bmatrix}$$

Clearly Theorem 5.1 follows from the following:

Theorem 5.3. *Let M, N be two A -modules and let $M = M_1^{a_1} \oplus \cdots \oplus M_s^{a_s}$ and $N = N_1^{b_1} \oplus \cdots \oplus N_r^{b_r}$ with M_i, N_j indecomposable. Assume $M_i \not\cong N_j$ for all i, j . Also assume $M_i \not\cong M_j$ for $i \neq j$ and $N_i \not\cong N_j$ for $i \neq j$. Then*

$$\text{rad End}_A(M \oplus N) = \begin{bmatrix} \text{rad End}_A(M) & \text{Hom}_A(N, M) \\ \text{Hom}_A(M, N) & \text{rad End}_A(N) \end{bmatrix}$$

We need the following

Lemma 5.4. *(with hypotheses as in 5.3) Let $f \in \text{Hom}_A(M, N)$ and let $g \in \text{Hom}_A(N, M)$. Then $g \circ f \in \text{rad End}_A(M)$ and $f \circ g \in \text{rad End}_A(N)$.*

Proof. We induct on s , the number of factors of M .

We first prove when $s = 1$. So $M = M_1^{a_1}$. It suffices to prove the result when N is indecomposable. If $N = D_1 \oplus D_2$ and assume the result is known for D_1 and D_2 . Let $f: M \rightarrow N$ and $g: N \rightarrow M$. Write $f = (f_1, f_2)$ and $g = g_1 + g_2$ where $f_i: M \rightarrow D_i$ and $g_i: D_i \rightarrow M$ for $i = 1, 2$. Then $g \circ f = g_1 \circ f_1 + g_2 \circ f_2$. By our assumption, $g_i \circ f_i \in \text{rad End}(M)$. So $g \circ f \in \text{rad End}(M)$.

We first consider the case when $a_1 = 1$. So let $f: M_1 \rightarrow N$ and let $g: N \rightarrow M_1$. If $g \circ f \notin \text{rad End}(M_1)$ then it is invertible as M_1 is indecomposable. It follows that $M_1 \cong N$ (see the proof in [8, Chapter X, Lemma 7.6]) This is a contradiction.

Now assume $a_1 \geq 2$. Let $f: M_1^{a_1} \rightarrow N$ and let $g: N \rightarrow M_1^{a_1}$. Write $f = f_1 + \cdots + f_{a_1}$ and $g = [g_1, \cdots, g_{a_1}]^{tr}$ where $f_i: M_1 \rightarrow N$ and $g_i: N \rightarrow M_1$ for all i . Then notice $g \circ f = [g_i f_j]$. By the previous case we get that $g_i f_j \in \text{rad End}(M_1)$. It follows that

$$g \circ f \in M_{a_1}(\text{rad End}(M_1)) = \text{rad}(M_{a_1}(\text{End}(M_1))) = \text{rad End}(M_1^{a_1}).$$

(for the first equality above see [7, p. 61].)

Assume the result for $s = c$. We prove it when $s = c + 1$. Let $M = M_1^{a_1} \oplus \cdots \oplus M_c^{a_c} \oplus M_{c+1}^{a_{c+1}}$. Set $D = M_1^{a_1} \oplus \cdots \oplus M_c^{a_c}$. Then $M = D \oplus M_{c+1}^{a_{c+1}}$. Let $f: M \rightarrow N$ and $g: N \rightarrow M$. Write $f = [f_1, f_2]$ and $g = [g_1, g_2]^{tr}$ where $f_1: D \rightarrow N$ and $f_2: M_{c+1}^{a_{c+1}} \rightarrow N$ and $g_1: N \rightarrow D$ and $g_2: N \rightarrow M_{c+1}^{a_{c+1}}$. Then

$$g \circ f = \begin{bmatrix} g_1 \circ f_1 & g_1 \circ f_2 \\ g_2 \circ f_1 & g_2 \circ f_2 \end{bmatrix}$$

By induction hypotheses we have $g_1 \circ f_1 \in \text{rad End}(D)$ and $g_2 \circ f_2 \in \text{rad End}(M_{c+1}^{a_{c+1}})$.

Set

$$\begin{aligned} \xi_1 &= \begin{bmatrix} g_1 \circ f_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \xi_2 = \begin{bmatrix} 0 & g_1 \circ f_2 \\ 0 & 0 \end{bmatrix} \\ \xi_3 &= \begin{bmatrix} 0 & 0 \\ g_2 \circ f_1 & 0 \end{bmatrix} \quad \text{and} \quad \xi_4 = \begin{bmatrix} 0 & 0 \\ 0 & g_2 \circ f_2 \end{bmatrix} \end{aligned}$$

As $g \circ f = \xi_1 + \xi_2 + \xi_3 + \xi_4$ it suffices to show that $\xi_i \in \text{rad End}(M)$ for each i .

Let $\phi = [\phi_{ij}] \in \text{End}(M)$. Then notice

$$1 - \phi\xi_1 = \begin{bmatrix} 1 - \phi_{11} \circ g_1 \circ f_1 & 0 \\ -\phi_{21} \circ g_1 \circ f_1 & 1 \end{bmatrix}$$

Notice $\phi_{11} \circ g_1 \circ f_1 \in \text{rad End}(D)$. So $1 - \phi_{11} \circ g_1 \circ f_1$ is invertible in $\text{End}(D)$. It follows that $1 - \phi\xi_1$ is invertible. So $\xi_1 \in \text{rad End}(M)$. Similarly $\xi_4 \in \text{rad End}(M)$.

We now prove $\xi_2 \in \text{rad End}(M)$. Set $\theta = g_1 \circ f_2$. Notice

$$1 - \phi\xi_2 = \begin{bmatrix} 1 & -\phi_{11} \circ \theta \\ 0 & 1 - \phi_{21} \circ \theta \end{bmatrix}$$

Also note that $\phi_{21}: D \rightarrow M_{c+1}^{a_{c+1}}$ and $\theta: M_{c+1}^{a_{c+1}} \rightarrow D$. So by induction hypotheses we have that $\phi_{21} \circ \theta \in \text{rad End}(M_{c+1}^{a_{c+1}})$. It follows that $1 - \phi\xi_2$ is invertible in $\text{End}(M)$. So $\xi_2 \in \text{rad End}(M)$. Similarly $\xi_3 \in \text{rad End}(M)$. \square

We now give

Proof of Theorem 5.3. Set

$$\mathfrak{q} = \begin{bmatrix} \text{rad End}_A(M) & \text{Hom}_A(N, M) \\ \text{Hom}_A(M, N) & \text{rad End}_A(N) \end{bmatrix}$$

We first prove that \mathfrak{q} is a two sided ideal contained in $\text{rad End}(M \oplus N)$. Let $\xi = [\xi_{ij}] \in \mathfrak{q}$. Let $\phi = [\phi_{ij}] \in \text{End}(M)$. Then

$$\phi \circ \xi = \begin{bmatrix} \phi_{11}\xi_{11} + \phi_{12}\xi_{21} & \phi_{11}\xi_{12} + \phi_{12}\xi_{22} \\ \phi_{21}\xi_{11} + \phi_{22}\xi_{21} & \phi_{21}\xi_{12} + \phi_{22}\xi_{22} \end{bmatrix}$$

By Lemma 5.4 we have that $\phi_{12}\xi_{21} \in \text{rad End}(M)$ and $\phi_{21}\xi_{12} \in \text{rad End}(N)$. It follows that $\phi \circ \xi \in \mathfrak{q}$. Similarly $\xi \circ \phi \in \mathfrak{q}$. Therefore \mathfrak{q} is an ideal in $\text{End}(M \oplus N)$.

We now show that $\mathfrak{q} \subseteq \text{rad End}(M \oplus N)$. Let $\xi = [\xi_{ij}] \in \mathfrak{q}$. Set

$$\xi_1 = \begin{bmatrix} \xi_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \xi_2 = \begin{bmatrix} 0 & \xi_{12} \\ 0 & 0 \end{bmatrix}$$

$$\xi_3 = \begin{bmatrix} 0 & 0 \\ \xi_{21} & 0 \end{bmatrix} \quad \text{and} \quad \xi_4 = \begin{bmatrix} 0 & 0 \\ 0 & \xi_{22} \end{bmatrix}$$

As $\xi = \xi_1 + \xi_2 + \xi_3 + \xi_4$. It suffices to show that each $\xi_i \in \text{rad End}(M \oplus N)$. This is similar to the proof in 5.4.

As $\text{End}(M \oplus N)/\mathfrak{q}$ is semi-simple and $\mathfrak{q} \subseteq \text{rad End}(M \oplus N)$ it follows that $\mathfrak{q} = \text{rad End}(M \oplus N)$ \square

6. PROOF OF THEOREM 1.7

In this section we give a proof of Theorem 1.7. We restate it for the convenience of the reader.

Theorem 6.1. *Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat local homomorphism of Henselian local rings with fiber $F = B/\mathfrak{m}B$ regular local. Assume the residue fields $k = A/\mathfrak{m}$ and $l = B/\mathfrak{n}$ are perfect. Let M be a finitely generated A -module such that $\Gamma = \text{Hom}_A(M, M)$ has finite global dimension. Then*

$$\text{gldim } \Gamma \otimes_A B \leq \text{gldim } \Gamma + \dim F.$$

Proof. We first consider the case $M = M_1^{a_1} \oplus \cdots \oplus M_s^{a_s}$ where M_i are distinct absolutely indecomposable A -modules. Set $\mathfrak{r} = \text{rad } \Gamma$ and $\mathfrak{q} = \text{rad}(\Gamma \otimes B)$. By 2.3 we get that $\mathfrak{r} \otimes B \subseteq \mathfrak{q}$. Let $x_1, \dots, x_c \in \mathfrak{n}$ be such that their images in F minimally generate the maximal ideal of F . By 2.1 we get that $(\mathbf{x})(\Gamma \otimes B) \subseteq \mathfrak{q}$.

By Theorem 5.1 we get that Γ/\mathfrak{r} is a direct product of matrix rings over k . Therefore $(\Gamma/\mathfrak{r}) \otimes B$ is a direct product of matrix rings over F . It follows that $\mathfrak{q} = \mathfrak{r} \otimes B + (\mathbf{x})(\Gamma \otimes B)$.

By [11, p. 177], \mathbf{x} is a B -regular sequence. Also note that $\mathbf{x} \in Z(\Gamma \otimes B)$ the center of $\Gamma \otimes B$. Let $\mathbf{K} = \mathbf{K}(\mathbf{x}, B)$ be the Koszul complex of B with respect to \mathbf{x} . As $\Gamma/\mathfrak{r} \otimes B$ is a direct product of matrix rings over F we get that

$$\mathbf{C} = \mathbf{K} \otimes_B \left(\frac{\Gamma}{\mathfrak{r}} \otimes B \right)$$

is also acyclic with zeroth homology group $(\Gamma \otimes B)/\mathfrak{q}$.

We now note that $\text{projdim } \mathbf{C}_i = \text{projdim } \Gamma/\mathfrak{r} \otimes B$ for each module in the complex \mathbf{C} . Also note that $\text{projdim } \Gamma/\mathfrak{r} \otimes B \leq \text{projdim } \Gamma/\mathfrak{r} = \text{gldim } \Gamma$, here the second equality holds since Γ is semi-perfect with radical \mathfrak{r} . It follows that $\text{projdim}(\Gamma \otimes B)/\mathfrak{q} \leq c + \text{gldim } \Gamma$. We note that $\Gamma \otimes B$ is semi-perfect with radical \mathfrak{q} . So

$$\text{gldim } \Gamma \otimes B = \text{projdim}(\Gamma \otimes B)/\mathfrak{q} \leq c + \text{gldim } \Gamma.$$

Thus we have proved the result in this case.

Now we consider the general case. By 4.10 there exists a commutative diagram of flat local maps of Henselian rings

$$\begin{array}{ccc} A & \xrightarrow{\eta} & A' \\ \downarrow \phi & & \downarrow \phi' \\ B & \xrightarrow{\delta} & B' \end{array}$$

such that

- (1) (A', \mathfrak{m}') is a finite flat extension of A with $\mathfrak{m}A' = \mathfrak{m}'$.
- (2) $M \otimes A'$ is a direct sum of absolutely indecomposable A' -modules.
- (3) (B', \mathfrak{n}') is a finite flat extension of B with $\mathfrak{n}B' = \mathfrak{n}'$.

(4) The fiber F' of ϕ' is also regular.

Notice $\dim F' = \dim F = c$. Let $t = \text{gldim } \Gamma$. By 2.5 we have that $\text{gldim } \Gamma \otimes_A A' = \text{gldim } \Gamma = t$. By our previous case we have that

$$\text{gldim}(\Gamma \otimes_A A') \otimes_{A'} B' \leq t + c.$$

We now note that

$$(\Gamma \otimes_A A') \otimes_{A'} B' \cong \Gamma \otimes_A B' \cong (\Gamma \otimes_A B) \otimes_B B'.$$

By 2.5 we get that

$$\text{gldim } \Gamma \otimes_A B = \text{gldim}(\Gamma \otimes_A B) \otimes_B B' \leq t + c.$$

□

7. PROOF OF THEOREMS 1.2 AND 1.5

We first give

Proof of Theorem 1.2. We first note that if L is a field of characteristic 0 or p , the ring $L[X_1, \dots, X_n]/(f)$ is regular.

(i) By [11, p. 177] we get that T is a flat extension of A . So the map $\phi: A \rightarrow A(f)$ is flat and local. Also note the fiber of ϕ is $k[X_1, \dots, X_n]/(f)$ localized at a maximal ideal. In particular it is regular local. Thus $A(f)$ is Cohen-Macaulay. Furthermore $A(f)$ is Gorenstein if A is.

(ii) Clearly $A(f)$ is excellent. The residue field of $A(f)$ is the residue field of $k[X_1, \dots, X_n]/(f)$ localized at a maximal ideal. In particular it is a finite extension of k and so it is perfect. It is also clear that $A(f)$ is of mixed characteristic.

As $A(f)$ is local, to prove that it is a normal domain it suffices to prove it is normal. As $A(f)$ is Cohen-Macaulay clearly it satisfies S_2 . Also as $A(f)$ is catenary it suffices to show that $A(f)_P$ is regular for every prime ideal P of height one.

Let P be a prime ideal in $A(f)$ of height one. Let $\mathfrak{q} = P \cap A$. Note that we have a flat local map $\psi: A_{\mathfrak{q}} \rightarrow A(f)_P$. Let F be the fiber of ψ . We now note that

$$1 = \text{ht } P = \dim A(f)_P = \dim A + \dim F = \text{ht } \mathfrak{q} + \dim F.$$

Thus $\text{ht } \mathfrak{q} \leq 1$.

Case 1: $\text{ht } \mathfrak{q} = 0$. So $\mathfrak{q} = 0$. Let K be the quotient field of A . Then $A_{\mathfrak{q}} = K$. Also F is a localization of $K[X_1, \dots, X_n]/(f)$ and so is regular. It follows that $A(f)_P$ is regular in this case.

Case 2: $\text{ht } \mathfrak{q} = 1$. This implies that $\dim F = 0$. Let $\kappa(\mathfrak{q})$ be the residue field of $A_{\mathfrak{q}}$. Then note that F is $\kappa(\mathfrak{q})[X_1, \dots, X_n]/(f)$ localized at a minimal prime. We note that $\kappa(\mathfrak{q})$ is either of characteristic zero or p . As observed earlier $\kappa(\mathfrak{q})[X_1, \dots, X_n]/(f)$ is a regular ring. So F is a field. As A is normal, $A_{\mathfrak{q}}$ is regular. It follows that $A(f)_P$ is regular.

Thus $A(f)$ satisfies R_1 . So $A(f)$ is normal. As $A(f)$ is local we get that $A(f)$ is a normal domain.

(iii) Set $B = A(f)$. Let $\Gamma = \text{End}_A(M)$ be a NCCR of A . It is clear that $M \otimes_A B$ is a reflexive A -module and that $\Lambda = \Gamma \otimes B$ is maximal Cohen-Macaulay as an B -module. We also note that B has a canonical module, [2, 3.3.14] Thus it suffices to prove $\text{gldim } \Lambda = \dim B$, [5, 2.17]. By 2.5 we may complete B . Thus it suffices to prove $\text{gldim } \Lambda \otimes \widehat{B} = \dim B$. By [11, 22.4] the map $\phi: A \rightarrow B$ extends to a flat map $\widehat{\phi}: \widehat{A} \rightarrow \widehat{B}$. Also note that

$$\Lambda \otimes_B \widehat{B} = (\Gamma \otimes_A \widehat{A}) \otimes_{\widehat{A}} \widehat{B}.$$

By Theorem 1.7 we get that

$$\begin{aligned} \text{gldim } \Lambda \otimes_B \widehat{B} &\leq \text{gldim } \Gamma \otimes_A \widehat{A} + \dim \widehat{B} - \dim \widehat{A} \\ &= \text{gldim } \Gamma + \dim B - \dim A \\ &= \dim B \end{aligned}$$

Thus $\text{gldim } \Lambda \leq \dim B$, by 2.4. As Λ is maximal Cohen-Macaulay we always have $\text{gldim } \Lambda \geq \dim B$, see [10, F.1]. Thus $\text{gldim } \Lambda = \dim B$. It follows that Λ is a NCCR for B . \square

7.1. A proof of Theorem 1.5 can be given along the same lines as above. The only thing to note that for any prime \mathfrak{q} in A the residue field $\kappa(\mathfrak{q})$ of $A_{\mathfrak{q}}$ contains k . So $\kappa(\mathfrak{q})[X_1, \dots, X_n]/(f)$ is a regular ring.

8. PROOF OF THEOREM 1.9

In this section we give

Proof of Theorem 1.9. (i) Let K be the quotient field of A . We first prove that f is irreducible in $K[X_1, \dots, X_n]$. Let \overline{K} be the algebraic closure of K . As $k \subseteq K$ we get that $\overline{k} \subseteq \overline{K}$. As \tilde{f} is smooth in $\mathbb{P}^n(\overline{k})$ we get that \tilde{f} and its partial derivatives do not have a common zero in \mathbb{P}^n . Therefore $(X_0, X_1, \dots, X_n) = \sqrt{(\tilde{f}, J(\tilde{f}))}$ in $\overline{k}[X_0, \dots, X_n]$. It follows that $(X_0, X_1, \dots, X_n) = \sqrt{(\tilde{f}, J(\tilde{f}))}$ in $\overline{K}[X_0, \dots, X_n]$. Therefore \tilde{f} is irreducible in $\overline{K}[X_0, X_1, \dots, X_n]$. It follows that f is irreducible in $\overline{K}[X_1, \dots, X_n]$ and hence it is irreducible in $K[X_1, \dots, X_n]$.

Claim 1: $T(f)$ is a domain.

We assert that $(f)K[X_1, \dots, X_n] \cap A[X_1, \dots, X_n] = (f)A[X_1, \dots, X_n]$. If this assertion is proved then $T(f)$ will be a subring of $k[X_1, \dots, X_n]/(f)$ and so a domain.

Let $\xi \in (f)K[X_1, \dots, X_n] \cap A[X_1, \dots, X_n]$. Then $\xi = fg$ for some $g \in K[X_1, \dots, X_n]$. Clearing denominators of g we get that there exists $a \in A$ and $w \in A[X_1, \dots, X_n]$ such that $\xi a = fw$. We prove that a divides all the coefficients of w . Let P be a height one prime in A . Then A_P is a DVR, so in particular a

UFD. Let $R = A_P[X_1, \dots, X_{n-1}]$ and let L be the quotient field of R . We may assume that X_n appears as a term in f . So $f \notin L$. Note L is also the quotient field of $K[X_1, \dots, X_{n-1}]$. Note that by Gauss Lemma, f is irreducible in $L[X_n]$ as the content of f is one. So again by Gauss Lemma we get that f is irreducible in $R[X_n] = A_P[X_1, \dots, X_n]$. It follows that if c is a coefficient of w then $(c/a)_P \in A_P$. But A is a normal domain. So

$$\bigcap_{\text{ht } \mathfrak{q}=1} A_{\mathfrak{q}} = A.$$

It follows that $c/a \in A$. Thus $\xi \in (f)A[X_1, \dots, X_n]$.

By [11, p. 177], f is a non-zero divisor of $A[X_1, \dots, X_n]$ and $T(f)$ is a flat extension of A . Also note that $\dim T(f) = \dim A + n - 1$. Let \mathfrak{n} be a maximal ideal of $T(f)$.

Claim-2 $\mathfrak{n} \cap A$ is a maximal ideal of A .

Clearly $T(f)$ is an affine ring. Also by Claim-1 we have that $T(f)$ is a domain. Thus $\text{ht } \mathfrak{n} = \dim T(f) = \dim A + n - 1$. Let $P = \mathfrak{n} \cap A$. We have a flat map $\psi: A_P \rightarrow T(f)_{\mathfrak{n}}$. Let F be the fiber of ψ . Then note that F is a localization of $\kappa(P)[X_1, \dots, X_n]/(f)$. As the content of f is one we get that f is a non-zero divisor in $\kappa(P)[X_1, \dots, X_n]$. So $\dim F \leq n - 1$. By the dimension formula for flat extensions it follows that $\text{ht } P \geq \dim A$. So P is a maximal ideal of A .

Claim 3: $T(f)$ is Cohen-Macaulay. Also if A is Gorenstein then so is $T(f)$.

We first note that $k[X_1, \dots, X_n]/(f)$ is a regular ring. Let \mathfrak{n} be a maximal ideal of $T(f)$. Then by Claim 2 we get that $\mathfrak{m} = \mathfrak{n} \cap A$ is a maximal ideal of A . We have a flat extension $\psi: A_{\mathfrak{m}} \rightarrow T(f)_{\mathfrak{n}}$. Note that $\kappa(\mathfrak{m}) = A/\mathfrak{m}$ is a finite extension of k . As k is perfect we get that $D = \kappa(\mathfrak{m})[X_1, \dots, X_n]/(f)$ is regular. The fiber F of ψ is a localization of D and so is regular. Thus $T(f)_{\mathfrak{n}}$ is Cohen-Macaulay and is Gorenstein if A is.

The assertion that $T(f)$ is normal follows exactly as in the argument in the proof of Theorem 1.2.

(ii) Let $\Gamma = \text{End}_A(M)$ be a NCCR of A . Clearly $M \otimes T(f)$ is a reflexive $T(f)$ -module. Furthermore $\Lambda = \Gamma \otimes_A T(f)$ is maximal Cohen-Macaulay as a $T(f)$ -module. The ring $T(f)$ has a canonical module [6, 18.21]. So it suffices to prove that $\text{gldim } \Gamma_{\mathfrak{n}} = \dim T(f)_{\mathfrak{n}}$ for all maximal ideals \mathfrak{n} of $T(f)$. We note that $\mathfrak{n} \cap A = \mathfrak{m}$ a maximal ideal of A (by claim 2). Also as A and $T(f)$ are affine domains over k we get that the residue fields of \mathfrak{m} and \mathfrak{n} are finite extensions of k and so perfect. The proof of the assertion $\text{gldim } \Gamma_{\mathfrak{n}} = \dim T(f)_{\mathfrak{n}}$ follows exactly as in the case of Theorem 1.2. \square

REFERENCES

- [1] N. Bourbaki, *Algebré commutative. ix: Anneaux locaux réguliers complets*, Masson, Paris, 1983.

- [2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Revised Edition, vol. 39, Cambridge studies in advanced mathematics, Cambridge University Press, Cambridge, 1997.
- [3] A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique*, Chap. III(part1), Inst. Hautes Etudes Sci. Publ. Math. 24, 1965.
- [4] O. Iyama and M. Wemyss, *On the noncommutative Bondal-Orlov conjecture*, J. Reine Angew. Math. 683 (2013), 119-128.
- [5] O. Iyama and M. Wemyss, *Maximal Modifications and Auslander-Reiten Duality for Non-isolated Singularities* Invent. Math(to appear), eprint: arXiv:1007.1296v4
- [6] S. Iyengar et. al, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, 87. American Mathematical Society, Providence, RI, 2007. xviii+282 pp.
- [7] T. Y. Lam, *A first course in noncommutative rings*, Second edition. Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 2001.
- [8] S. Lang, *Algebra*, Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.
- [9] G. J. Leuschke, *Endomorphism rings of finite global dimension*, Canad. J. Math. 59 (2007), no. 2, 332-342.
- [10] G. J. Leuschke, *Non-commutative crepant resolutions: scenes from categorical geometry*, Progress in commutative algebra 1, 293-361, de Gruyter, Berlin, 2012.
- [11] H. Matsumura, *Commutative ring theory*, Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1989.
- [12] T. J. Puthenpurakal, *On two dimensional mixed characteristic rings of finite Cohen Macaulay type*, Preprint: arXiv:1404.6939
- [13] M. Ramras, *Maximal orders over regular local rings of dimension two*, Trans. Amer. Math. Soc. 142 1969 457-479.
- [14] M. van den Bergh, *Non-commutative crepant resolutions*, The legacy of Niels Henrik Abel, 749-770, Springer, Berlin, 2004.
- [15] R. Wiegand, *Local rings of finite Cohen-Macaulay type*, J. Algebra 203 (1998), no. 1, 156-168.

DEPARTMENT OF MATHEMATICS, IIT BOMBAY, POWAI, MUMBAI 400 076

E-mail address: tputhen@math.iitb.ac.in